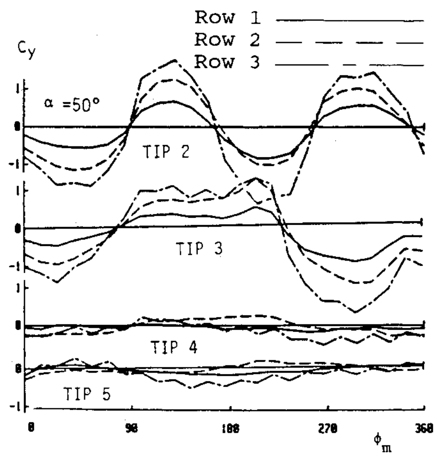

Fig. 3 Ogive tips 1, 3, 4, and 5: Variation of sectional side-force coefficient with model roll position.

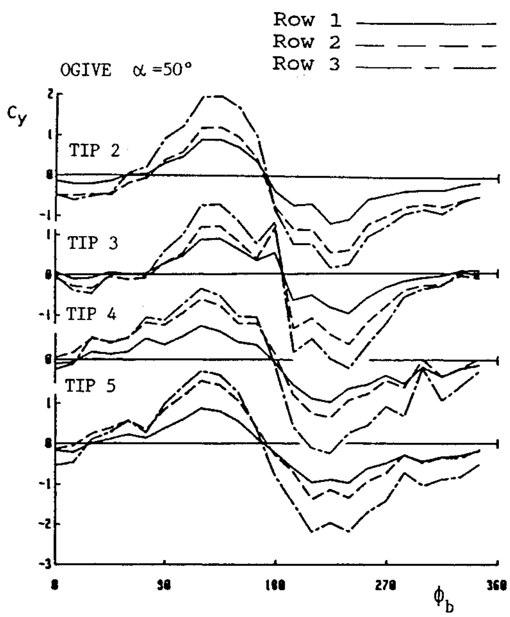

Fig. 4 Effect of tip shape on variation of sectional side-force coefficient with circumferential bead position.

bead was placed at varying azimuthal locations in 15-deg increments, all at a fixed axial position of 1.0 in., which corresponded to the tip/body junction. The bead was ⅛ in. high with a diameter of ⅛ in. Note that both the shape of the side-force distributions and the maximum values were nearly identical for all tips, including the blunter ones. The large side forces observed when the bead was nearly 45 deg from the windward ray indicates that blunt tips may produce as much flowfield asymmetry as sharper tips, if equivalent model asymmetry is present.

Previous research by the authors[3] has shown that the effect of a singular roughness element is proportional to the element's size relative to the dimensions of the body where the roughness occurs. Therefore, reductions in side force with bluntness are largely the result of three factors. First, surface imperfections on a blunt tip are smaller relative to the tip diameter at which they occur than the same disturbances would be on the apex of a sharper tip. Second, machining processes are less likely to produce irregularities on a sturdy blunt tip than on a relatively flimsy sharp one. Third, blunter tips are more difficult to damage and, therefore, more likely to keep their shape. Microscopic photographs of some of the tips tested, showing machining imperfections, can be seen in Ref. 2.

References

[1]Hunt, B. L., "Asymmetric Vortex Forces and Wakes on Slender Bodies," AIAA Paper 82-1336, Aug. 1982.

[2]Moskovitz, C. A., Hall, R. M., and DeJarnette, F. R., "Effects of Nose Bluntness, Roughness and Surface Perturbations on the Asymmetric Flow Past Slender Bodies at Large Angles of Attack," AIAA Paper 89-2236, Aug. 1989.

[3]Moskovitz, C. A., Hall, R. M., and DeJarnette, F. R., "Effects of Surface Perturbations on the Asymmetric Vortex Flow Over a Slender Body," AIAA Paper 88-0483, Jan. 1988.

Accurate Method for Calculating Initial Development of Vortex Sheets

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Introduction

IN this Note, the dynamics of a planar, infinitesimally thin, finite-span vortex sheet placed at time $t=0$ in an otherwise unbounded stationary fluid is studied for small times, $t>0$. The solution is expressed as a power series in time whose coefficients are calculated accurately by a proper numerical evaluation of the accompanying Cauchy integrals. The method does not discretize the vortex sheet and is the numerical counterpart of the semianalytical method of Ref. 1.

The more difficult problem—when the vortex sheet is curved at $t=0$—has been addressed in Ref. 2 using a time-marching scheme. This method is computationally expensive because very small time steps are required to maintain accuracy. In comparison, for the initially plane vortex sheet, the method presented herein is at least three orders of magnitude faster. It also serves the long-felt need to determine rapidly the initial development of an aircraft wing wake to engineering accuracy, for which the two-dimensional vortex sheet is considered a reasonable model.[3,4]

Problem Statement

Consider a two-dimensional free vortex sheet whose vorticity vector is perpendicular to the x,y plane and whose Trefftz plane cross section at some instant $t=0$ spans the line $-1\leq x\leq 1$, $y=0$. Let the sheet vorticity distribution be $G(x)$. At any instant $t>0$, let $[X(x,t),\ Y(x,t)]$ be the coordinates of the fluid element originally located at $(x,0)$ at $t=0$. The convective motion of the vortex sheet is then governed by

$$u(x,t)=\frac{\mathrm{d}X}{\mathrm{d}t}=-(\tfrac{1}{2}\pi)\int_{-1}^{1}S^{-1}(x,\xi,t)\times G(\xi)\big[Y(x,t)-Y(\xi,t)\big]\,\mathrm{d}\xi \tag{1a}$$

$$v(x,t)=\frac{\mathrm{d}Y}{\mathrm{d}t}=(\tfrac{1}{2}\pi)\int_{-1}^{1}S^{-1}(x,\xi,t)\times G(\xi)\big[X(x,t)-X(\xi,t)\big]\,\mathrm{d}\xi \tag{1b}$$

where use has been made of the fact that, in an incompressible, inviscid, two-dimensional fluid flow, vorticity is a passively convected scalar. Furthermore, $u(x,t)$ and $v(x,t)$ are the x and y components, respectively, of the sheet convection veloc-

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ity, and

$$S(x,\xi,t) = [X(x,t) - X(\xi,t)]^2 + [Y(x,t) - Y(\xi,t)]^2 \quad (2)$$

For the nomenclature, see Fig. 1. The problem is, for a given $G(x)$, $X(x,0)$, and $Y(x,0)$, determine $X(x,t)$ and $Y(x,t)$.

Equation (1) is valid everywhere, including points on the vortex sheet. On the vortex sheet, the integral is to be interpreted in the Cauchy principal value sense.[1]

The early evolution of the vortex sheet may be obtained by expressing $X(x,t)$ and $Y(x,t)$ in the following Taylor series:

$$X(x,t) = x + \sum_{n=1} \left[\left(\frac{d^n}{dt^n} \right) X(x,0) \right] \frac{t^n}{n!} \quad (3a)$$

$$Y(x,t) = \sum_{n=1} \left[\left(\frac{d^n}{dt^n} \right) Y(x,0) \right] \frac{t^n}{n!} \quad (3b)$$

where the higher derivatives of X and Y may be obtained from Eq. (1) by an appropriate number of differentiations, the first few of which are given in the Appendix. Following a lemma due to Lighthill,[5] it can be shown that, if $(1-x^2)^{1/2} G(x)$ is regular in $-1 \le x \le 1$, then all of the Cauchy integrals appearing in Eq. (3) would also be regular in $-1 \le x \le 1$. This implies that $X(x,t)$ and $Y(x,t)$ can be expressed as polynomials in x. For any other $G(x)$, the integrals are singular at $x = \pm 1$. We shall, therefore, restrict our attention to only those $G(x)$ for which $(1-x^2)^{1/2} G(x)$ is regular in $-1 \le x \le 1$.

Numerical Scheme

For a $G(x)$ that satisfies Lighthill's lemma, it can be shown that the integrals in Eq. (3) have the generic form

$$I(x) = \int_{-1}^{1} (x-\xi)^{-1} h(\xi)\, d\xi \quad (4)$$

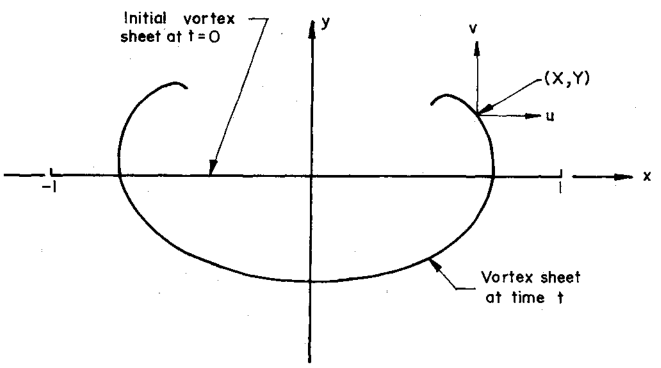

Fig. 1 Vortex sheet nomenclature.

Table 1 Comparison of numerically calculated vortex sheet shape at $t = 0.4$ with Schwartz' analytical time-series solution. Planar vortex sheet at $t = 0$ is described by $G(x) = 3x(1-x^2)^{1/2}$; $-1 \le x \le 1$, $y = 0$

	at $t = 0.4$			
	Schwartz		Present	
x	X	Y	X	Y
0.0000	0.0000	−0.2844	0.0000	−0.2844
±0.1564	±0.1803	−0.2697	±0.1803	−0.2697
±0.3090	±0.3495	−0.2273	±0.3495	−0.2273
±0.4540	±0.4983	−0.1625	±0.4983	−0.1625
±0.5878	±0.6200	−0.0828	±0.6200	−0.0828
±0.7071	±0.7106	0.0020	±0.7106	0.0020
±0.8090	±0.7697	0.0821	±0.7697	0.0821
±0.8910	±0.8006	0.1489	±0.8006	0.1489
±0.9511	±0.8111	0.1972	±0.8111	0.1972
±0.9877	±0.8114	0.2256	±0.8114	0.2256

where $(1-\xi^2)^{1/2} h(\xi)$ will be regular in $-1 \le \xi \le 1$, and, hence, $I(x)$ will be regular in $-1 \le x \le 1$. These integrals are evaluated using Stark's algorithm[6]:

$$I(x_j) = \int_{-1}^{1} (\xi - x_j)^{-1} w(\xi) f(\xi)\, d\xi = \sum_{i=1}^{N} e_i w(\xi_i) f(\xi_i) / (\xi_i - x_j) \quad (5)$$

where[7]

$$h(\xi) = w(\xi) f(\xi)$$

$$w(\xi) = (1-\xi^2)^{-1/2}$$

$$f(\xi) = \text{a regular function of } \xi, \ = -1 \le \xi \le 1$$

$$e_i = \pi (1-\xi_i^2)^{1/2} / N$$

$$\xi_i = \cos[(2i-1)\pi/2N]; \qquad i = 1,2,\ldots,N$$

$$x_j = \cos(j\pi/N); \qquad j = 1,2,\ldots,N-1$$

$I(x)$ is calculated exactly at the points $x = x_j$ if $f(\xi)$ is a polynomial of degree $\le 2N$. The Stark algorithm (which is actually a Gaussian quadrature rule extended to Cauchy integrals), therefore, provides accuracy with economy. However, even for moderate values of N, say, $N > 4$, the computations must be done in at least double precision on a digital computer to minimize roundoff errors.

Numerical Example

Schwartz provides an analytical time-series solution for the following example (see also Ref. 8).

For $G(x)$ given by

$$(1-x^2)^{1/2} G(x) = 3x(1-x^2), \quad -1 \le x \le 1, \quad y = 0 \text{ at } t = 0$$

the solution is

$$X(x,t) = x + x[(9/8) - (9/4)x^2] t^2 - x[(135/128) - (135/32)x^2 + (27/8)x^4] t^4 + x[(8343/5120) - (31{,}509/2560)x^2 + (4131/160)x^4 - (81/5)x^6] t^6 + \cdots \quad (6a)$$

$$Y(x,t) = -[(3/4) - (3/2)x^2] t + [(9/32) - (9/8)x^4] t^3 - [(81/256) - (81/128)x^2 - (81/32)x^4 + (567/160)x^6] t^5 + [(7047/14{,}336) - (243/112)x^2 - (73{,}629/17{,}920)x^4 + (7533/320)x^6 - (90{,}639/4480)x^8] t^7 + \cdots \quad (6b)$$

Equation (3) was evaluated numerically by retaining the same number of terms in t as noted earlier in Eq. (6). A sample result is shown in Table 1. As expected, correspondence is achieved.

Conclusions

A fast and accurate numerical method for calculating the early evolutionary stages of an initially planar, finite-span vortex sheet has been found. The method may be used to determine wing downwash behind a wing, e.g., in the vicinity of the tail plane.

Appendix

Let the geometry of the plane vortex sheet at $t=0$ be given by

$$X(x,0)=x \qquad Y(x,0)=0, \qquad -1\le x\le 1$$

If we now define

$$\mathfrak{X}(x,\xi,t)\equiv X(x,t)-X(\xi,t)$$

$$\mathfrak{Y}(x,\xi,t)\equiv Y(x,t)-Y(\xi,t)$$

$$S(x,\xi,t)\equiv\left[X(x,t)-X(\xi,t)\right]^2+\left[Y(x,t)-Y(\xi,t)\right]^2$$

then the first few derivatives of X,Y at $t=0$ are given by

$$\frac{\mathrm{d}Y}{\mathrm{d}t}=(½\pi)\int_{-1}^{1}(x-\xi)^{-1}G(\xi)\,\mathrm{d}\xi$$

$$\frac{\mathrm{d}^2X}{\mathrm{d}t^2}=-(½\pi)\int_{-1}^{1}(x-\xi)^{-2}G(\xi)\left(\frac{\mathrm{d}\mathfrak{Y}}{\mathrm{d}t}\right)\mathrm{d}\xi$$

$$\frac{\mathrm{d}^3Y}{\mathrm{d}t^3}=-(½\pi)\int_{-1}^{1}(x-\xi)^{-3}G(\xi)\times\left[(x-\xi)\left(\frac{\mathrm{d}^2\mathfrak{X}}{\mathrm{d}t^2}\right)+2\left(\frac{\mathrm{d}\mathfrak{Y}}{\mathrm{d}t}\right)^2\right]\mathrm{d}\xi$$

$$\frac{\mathrm{d}^4X}{\mathrm{d}t^4}=-(½\pi)\int_{-1}^{1}(x-\xi)^{-4}G(\xi)\times\left[(x-\xi)^2\left(\frac{\mathrm{d}^3\mathfrak{Y}}{\mathrm{d}t^3}\right)-6(x-\xi)\left(\frac{\mathrm{d}\mathfrak{Y}}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}^2\mathfrak{X}}{\mathrm{d}t^2}\right)-6\left(\frac{\mathrm{d}\mathfrak{Y}}{\mathrm{d}t}\right)\right]\mathrm{d}\xi$$

$$\frac{\mathrm{d}^5Y}{\mathrm{d}t^5}=(½\pi)\int_{-1}^{1}(x-\xi)^{-5}G(\xi)\times\left[(x-\xi)^3\left(\frac{\mathrm{d}^4\mathfrak{X}}{\mathrm{d}t^4}\right)-(x-\xi)^2\left(\frac{\mathrm{d}^4S}{\mathrm{d}t^4}\right)-6(x-\xi)\left(\frac{\mathrm{d}^2\mathfrak{X}}{\mathrm{d}t^2}\right)\left(\frac{\mathrm{d}^2S}{\mathrm{d}t^2}\right)+6\left(\frac{\mathrm{d}^2S}{\mathrm{d}t^2}\right)^2\right]\mathrm{d}\xi$$

$$\frac{\mathrm{d}^6X}{\mathrm{d}t^6}=-(½\pi)\int_{-1}^{1}(x-\xi)^{-6}G(\xi)\times\left\{(x-\xi)^4\left(\frac{\mathrm{d}^5\mathfrak{Y}}{\mathrm{d}t^5}\right)-10(x-\xi)^2\left[\left(\frac{\mathrm{d}^3\mathfrak{Y}}{\mathrm{d}t^3}\right)\left(\frac{\mathrm{d}^2S}{\mathrm{d}t^2}\right)+(1/2)\left(\frac{\mathrm{d}\mathfrak{Y}}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}^4S}{\mathrm{d}t^4}\right)\right]+30\left(\frac{\mathrm{d}\mathfrak{Y}}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}^2S}{\mathrm{d}t^2}\right)^2\right\}\mathrm{d}\xi$$

$$\frac{\mathrm{d}^7Y}{\mathrm{d}t^7}=(½\pi)\int_{-1}^{1}(x-\xi)^{-7}G(\xi)\times\left\{(x-\xi)^5\left(\frac{\mathrm{d}^6\mathfrak{X}}{\mathrm{d}t^6}\right)-(x-\xi)^4\left(\frac{\mathrm{d}^6S}{\mathrm{d}t^6}\right)-15(x-\xi)^3\times\left[\left(\frac{\mathrm{d}^4\mathfrak{X}}{\mathrm{d}t^4}\right)\left(\frac{\mathrm{d}^2S}{\mathrm{d}t^2}\right)+\left(\frac{\mathrm{d}^2\mathfrak{X}}{\mathrm{d}t^2}\right)\left(\frac{\mathrm{d}^4S}{\mathrm{d}t^4}\right)\right]+30(x-\xi)^2\left(\frac{\mathrm{d}^4S}{\mathrm{d}t^4}\right)\left(\frac{\mathrm{d}^2S}{\mathrm{d}t^2}\right)+90(x-\xi)\left(\frac{\mathrm{d}^2\mathfrak{X}}{\mathrm{d}t^2}\right)\left(\frac{\mathrm{d}^2S}{\mathrm{d}t^2}\right)^2-90\left(\frac{\mathrm{d}^2S}{\mathrm{d}t^2}\right)^3\right\}\mathrm{d}\xi,\quad\text{etc.}$$

$$\frac{\mathrm{d}^{2n-1}X}{\mathrm{d}t^{2n-1}}=0,\qquad \frac{\mathrm{d}^{2n}Y}{\mathrm{d}t^{2n}}=0,\qquad n=1,2,\ldots$$

$$S(x,\xi,0)=(x-\xi)^2$$

$$\frac{\mathrm{d}S}{\mathrm{d}t}=0$$

$$\frac{\mathrm{d}^2S}{\mathrm{d}t^2}=2(x-\xi)\left(\frac{\mathrm{d}^2\mathfrak{X}}{\mathrm{d}t^2}\right)+2\left(\frac{\mathrm{d}\mathfrak{Y}}{\mathrm{d}t}\right)^2$$

$$\frac{\mathrm{d}^3S}{\mathrm{d}t^3}=0$$

$$\frac{\mathrm{d}^4S}{\mathrm{d}t^4}=2(x-\xi)\left(\frac{\mathrm{d}^4\mathfrak{X}}{\mathrm{d}t^4}\right)+6\left(\frac{\mathrm{d}^2\mathfrak{X}}{\mathrm{d}t^2}\right)^2+8\left(\frac{\mathrm{d}\mathfrak{Y}}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}^3\mathfrak{Y}}{\mathrm{d}t^3}\right)$$

$$\frac{\mathrm{d}^5S}{\mathrm{d}t^5}=0$$

$$\frac{\mathrm{d}^6S}{\mathrm{d}t^6}=2(x-\xi)\left(\frac{\mathrm{d}^6\mathfrak{X}}{\mathrm{d}t^6}\right)+30\left(\frac{\mathrm{d}^2\mathfrak{X}}{\mathrm{d}t^2}\right)\left(\frac{\mathrm{d}^4\mathfrak{X}}{\mathrm{d}t^4}\right)+12\left(\frac{\mathrm{d}\mathfrak{Y}}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}^5\mathfrak{Y}}{\mathrm{d}t^5}\right)+20\left(\frac{\mathrm{d}^3\mathfrak{Y}}{\mathrm{d}t^3}\right)^2$$

References

[1]Schwartz, L. W., "A Semi-analytic Approach to the Self-Induced Motion of Vortex Sheets," *Journal of Fluid Mechanics,* Vol. 111, Oct. 1981, pp. 475–490.

[2]Bera, R. K., "On the Self-Induced Motion of Vortex Sheets," *International Journal for Numerical Methods in Fluids,* Vol. 8, July 1988, pp. 759–768.

[3]Yates, J. E., "Calculation of Initial Vortex Roll-Up in Aircraft Wakes," *Journal of Aircraft,* Vol. 11, July 1974, pp. 397–400.

[4]Bera, R. K., "Calculation of Initial Vortex Roll-Up in Aircraft Wakes," *Journal of Aircraft,* Vol. 13, Jan. 1976, pp. 58–59.

[5]Lighthill, M. J., "A New Approach to Thin Aerofoil Theory," *The Aeronautical Quarterly,* Vol. III, Nov. 1951, pp. 193–210.

[6]Stark, V. J. E., "A Generalized Quadrature Formula for Cauchy Integrals," *AIAA Journal,* Vol. 9, Sept. 1971, pp. 1854–1855.

[7]Bera, R. K., "The Numerical Evaluation of Cauchy Integrals," *International Journal for Mathematical Education in Science and Technology,* Vol. 20, No. 5, 1989, pp. 763–776.

[8]Takami, H., "A Numerical Experiment with Discrete-Vortex Approximation with Reference to the Rolling Up of a Vortex Sheet," Dept. of Aeronautics and Astronautics, Stanford Univ., Stanford, CA, Rept. SUDAER 202, Sept. 1964.